

MTH250 - CALCULUS

Lecture No. 32

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These lecture notes are prepared to serve as handouts for the video lectures on the course *MTH-250 Calculus* of Virtual Campus, COMSATS Institute of Information Technology, Islamabad, Pakistan. The notes were derived from an extensive collection of notes and books. Most of the theoretical details as well as worked problems are from *Calculus*, by *H. Anton, I Bivens and S. Davis*, John Wiley & Sons, 10th edition, 2012.

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LECTURE 32

Triple integrals

32.1 Triple integrals

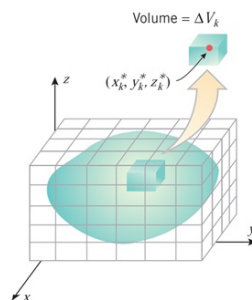


Figure 32.1: Partitioning of a solid

Consider a closed solid region G in an xyz -coordinate, enclosed by a box. Divide the box into n sub-boxes by planes parallel to the coordinate planes. Discard those sub-boxes that contain any points outside of G . Choose an arbitrary point in each of the remaining sub-boxes $P_k^*(x_k^*, y_k^*, z_k^*)$. Let ΔV_k denote the volume of the k^{th} sub-box. Now, form the product $f(x_k^*, y_k^*, z_k^*) \Delta V_k$. Then add the products for all sub-boxes to

obtain the Riemann sum

$$\sum_{k=1}^n f(x_k^*, y_k^*, z_k^*) \Delta V_k$$

Finally we repeat this process with more and more subdivisions in such a way that the length, width and height of each sub-box approach zero and $n \rightarrow \infty$

The limit

$$\iiint_G f(x, y, z) dV = \lim_{n \rightarrow \infty} \sum_{k=1}^n f(x_k^*, y_k^*, z_k^*) \Delta V_k$$

is called the triple integral of f over G .

Theorem 32.1 (Properties of triple integrals).

Triple integrals enjoy many properties of single and double integrals

1. $\iiint_G c f(x, y, z) dV = c \iiint_G f(x, y, z) dV$ (c a constant).
2. $\iiint_G [f(x, y, z) + g(x, y, z)] dV = \iiint_G f(x, y, z) dV + \iiint_G g(x, y, z) dV$
3. $\iiint_G [f(x, y, z) - g(x, y, z)] dV = \iiint_G f(x, y, z) dV - \iiint_G g(x, y, z) dV$
4. Moreover, if the region G is subdivided into two sub-regions G_1 and G_2

$$\iiint_G f(x, y, z) dV = \iiint_{G_1} f(x, y, z) dV + \iiint_{G_2} f(x, y, z) dV.$$

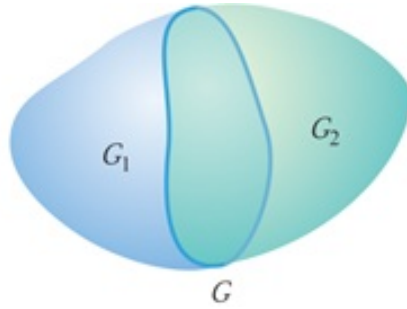


Figure 32.2: $G = G_1 + G_2$.

32.2 Evaluating triple integrals

Theorem 32.2 (Fubini's).

Let G be the rectangular box defined by the inequalities

$$a \leq x \leq b, \quad c \leq y \leq d, \quad k \leq z \leq l.$$

If f is continuous on the region G , then

$$\iiint_G f(x, y, z) dV = \int_a^b \int_c^d \int_k^l f(x, y, z) dx dy dz.$$

Moreover, the iterated integral on the right can be replaced with any of the five other iterated integrals that result by altering the order of integration.

Example 32.1. Evaluate the triple integral $\iiint 12xy^2z^3 dV$ over the rectangular G defined by the inequalities $-1 \leq x \leq 2$, $0 \leq y \leq 3$ and $0 \leq z \leq 2$.

Solution.

$$\begin{aligned} \iiint_G 12xy^2z^3 &= \int_{-1}^2 \int_0^3 \int_0^2 12xy^2z^3 dx dy dz \\ &= \int_{-1}^2 \int_0^3 \left[12xy^2z^3 \right]_{z=0}^2 dy dx = \int_{-1}^2 \int_0^3 48xy^2 dy dx \\ &= \int_{-1}^2 \left[16xy^3 \right]_{y=0}^3 dx = \int_{-1}^2 432x dx \\ &= 216x^2 \Big|_{-1}^2 = 648. \end{aligned}$$

□

Theorem 32.3. Let G be a simple xy -solid with upper surface $z = g_2(x, y)$ and lower surface $z = g_1(x, y)$, and let R be the projection of G on the xy -plane. If $f(x, y, z)$ is continuous on G , then

$$\iiint_G f(x, y, z) dV = \iint_R \left[\int_{g_1(x, y)}^{g_2(x, y)} f(x, y, z) dz \right] dA$$

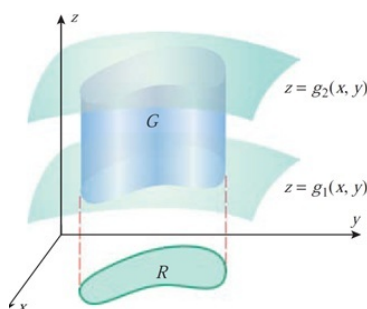


Figure 32.3: xy -solid with upper surface $z = g_2(x, y)$ and lower surface $z = g_1(x, y)$.

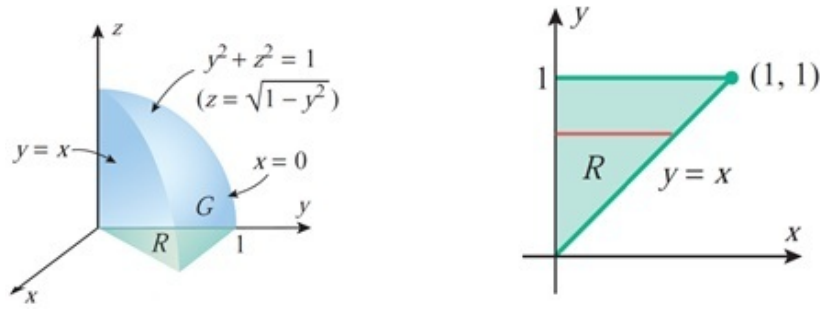


Figure 32.4: Wedge in the first octant that is cut from the cylindrical solid $y^2 + z^2 \leq 1$ by the planes $y = x$ and $x = 0$.

Example 32.2. Let G be the wedge in the first octant that is cut from the cylindrical solid $y^2 + z^2 \leq 1$ by the planes $y = x$ and $x = 0$. Evaluate $\iiint_G z dV$.

Solution.

The solid G and its projection R on the xy -plane. The upper surface of the solid is formed by the cylinder and the lower surface by the xy -plane. Since the portion of the cylinder $y^2 + z^2 = 1$ that lies above the xy -plane has the equation $z = \sqrt{1-y^2}$, and the xy -plane has the equation $z = 0$, it follows that

$$\iiint_G z dV = \iint_R \left[\int_0^{\sqrt{1-y^2}} z dz \right] dA.$$

□

For the double integral over R , the x - and y -integration can be performed in either order since R is both a type I and type II region. We will integrate with respect to x first. With this choice,

$$\begin{aligned} \iiint_G z dV &= \int_0^1 \int_0^y \left[\int_0^{\sqrt{1-y^2}} z dz \right] dA = \int_0^1 \int_0^y \left. \frac{1}{2} z^2 \right|_{z=0}^{\sqrt{1-y^2}} dx dy \\ &= \int_0^1 \int_0^y \frac{1}{2} (1-y^2) dx dy = \frac{1}{2} \int_0^1 (1-y^2) x \Big|_{x=0}^y dy \\ &= \int_0^1 (y - y^3) dy = \frac{1}{2} \left[\frac{1}{2} y^2 - \frac{1}{4} y^4 \right]_0^1 = \frac{1}{8} \end{aligned}$$

32.3 Volume as triple integral

In the special case when $f(x, y, z) = 1$

$$\text{volume of } G = \iiint_G dV = \lim_{n \rightarrow +\infty} \sum_{k=1}^n \Delta V_k$$

Example 32.3. Use a triple integral to find the volume of the solid within the cylinder $x^2 + y^2 = 9$ and between the planes $z = 1$ and $x + z = 5$.

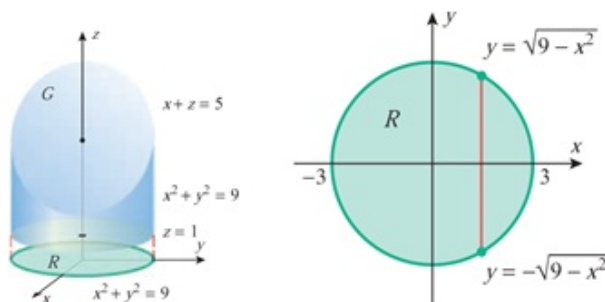


Figure 32.5: Solid within the cylinder $x^2 + y^2 = 9$ and between the planes $z = 1$ and $x + z = 5$.

Solution.

The lower surface of the solid is the plane $z = 1$ and the upper surface is the plane $x + z = 5$ or equivalently, $z = 5 - x$. Thus,

$$\text{Volume of } G = \iiint_G dV = \iint_R \left[\int_1^{5-x} dz \right] dA.$$

For the double integral over R , we will integrate with respect to y first. Thus,

$$\begin{aligned} \text{Volume of } G &= \int_{-3}^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_1^{5-x} dz dy dx = \int_{-3}^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} z \Big|_{z=1}^{5-x} dy dx \\ &= \int_{-3}^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} (4-x) dy dx = \int_{-3}^3 (8-2x) \sqrt{9-x^2} dx \\ &= 8 \int_{-3}^3 \sqrt{9-x^2} dx - \int_{-3}^3 2x \sqrt{9-x^2} dx \\ &= 8 \left(\frac{9}{2} \pi \right) - \int_{-3}^3 2x \sqrt{9-x^2} dx = 8 \left(\frac{9}{2} \pi \right) - 0 = 36\pi. \end{aligned}$$

□

32.4 Divergence theorem

Theorem 32.4 (Divergence).

Let G be a solid whose surface σ is oriented outward. If

$$\mathbf{F}(x, y, z) = f(x, y, z)\hat{i} + g(x, y, z)\hat{j} + h(x, y, z)\hat{k}$$

where f , g and h have continuous first partial derivatives on some open set containing G and if $\mathbf{n}$ is the outward unit normal on σ , then

$$\iint_{\sigma} \mathbf{F} \cdot \mathbf{n} dS = \iiint_G \operatorname{div} \mathbf{F} dV.$$

Divergence Theorem states that under the given conditions, the flux of $\mathbf{F}$ across the boundary surface σ of G is equal to the triple integral of the divergence of $\mathbf{F}$ over G .

The Divergence Theorem is sometimes called Gauss's Theorem after the great German mathematician Karl Friedrich Gauss (1777 – 1855). It is also known as Ostrogradsky's Theorem after the Russian mathematician Mikhail Ostrogradsky (1801 – 1862). He published this result in 1826.

Example 32.4. Using the Divergence theorem to find outward flux of the vector field $\mathbf{F}(x, y, z) = z\hat{k}$ across the sphere $x^2 + y^2 + z^2 = a^2$.

Solution. Let σ denote the outward-oriented spherical surface and G the region that it encloses. The divergence of the vector field is

$$\operatorname{div} \mathbf{F} = \frac{\partial z}{\partial z} = 1$$

so the flux across σ is

$$\Phi = \iint \mathbf{F} \cdot \mathbf{n} dS = \iiint_G dV = \text{volume of } G = \frac{4\pi a^3}{3}.$$

□

Example 32.5. Use the Divergence theorem to find the outward flux of the vector field

$$\mathbf{F}(x, y, z) = 2x\hat{i} + 3y\hat{j} + z^2\hat{k}$$

across the unit cube in the Figure 32.6.

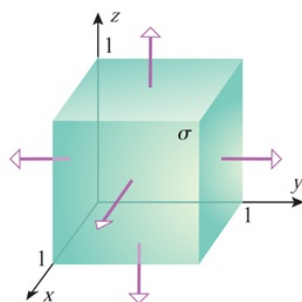


Figure 32.6: Unit cube.

Solution. Let σ denote the outward-oriented surface of the cube and G the region that it encloses. The divergence of the vector field is

$$\operatorname{div} \mathbf{F} = \frac{\partial}{\partial x}(2x) + \frac{\partial}{\partial y}(3y) + \frac{\partial}{\partial z}(z^2) = 5 + 2z$$

so the flux across σ is

$$\begin{aligned} \Phi &= \iint_{\sigma} \mathbf{F} \cdot \mathbf{n} dS = \iiint_G (5 + 2z) dV = \int_0^1 \int_0^1 \int_0^1 (5 + 2z) dz dy dx \\ &= \int_0^1 \int_0^1 [5z + z^2]_0^1 dy dx = \int_0^1 \int_0^1 6 dy dx = 6. \end{aligned}$$

□

Example 32.6. Use the Divergence theorem to find the outward flux of the vector field

$$\mathbf{F}(x, y, z) = x^3 \hat{i} + y^3 \hat{j} + z^2 \hat{k}$$

across the surface of the region that is enclosed by the circular cylinder $x^2 + y^2 = 9$, see Figure 32.7.

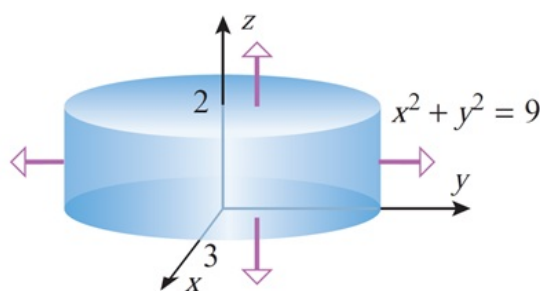


Figure 32.7: Circular cylinder $x^2 + y^2 = 9$.

Solution. Let σ denote the outward-oriented surface and G the region that it encloses. The divergence of the vector field is

$$\operatorname{div} \mathbf{F} = \frac{\partial}{\partial x}(x^3) + \frac{\partial}{\partial y}(y^3) + \frac{\partial}{\partial z}(z^2) = 3x^2 + 3y^2 + 2z$$

so the flux across σ is

$$\begin{aligned} \Phi &= \iint_{\sigma} \mathbf{F} \cdot \mathbf{n} dS = \iiint_G (3x^2 + 3y^2 + 2z) r dz dr d\theta = \int_0^{2\pi} \int_0^3 \int_0^2 (3r^2 + 2z) r dz dr d\theta \\ &= \int_0^{2\pi} \int_0^3 [3r^2 z + r z^2]_0^2 dr d\theta = \int_0^{2\pi} \int_0^3 (6r^3 + 4r) dr d\theta \\ &= \int_0^{2\pi} \left[\frac{3r^4}{2} + 2r^2 \right]_0^3 d\theta = \int_0^{2\pi} \frac{279}{2} d\theta = 279\pi. \end{aligned}$$

□